A trace identity and its applications to the theory of discrete integrable systems

1990 J. Phys. A: Math. Gen. 23 3903

(http://iopscience.iop.org/0305-4470/23/17/020)

J. Phys. A: Math. Gen. 23 (1990) 3903-3922. Printed in the UK

A trace identity and its applications to the theory of discrete integrable systems

Tu Gui-zhang

Dipartimento di Fisica, Università 'La Sapienza', Roma, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy

Received 17 October 1989, in final form 8 February 1990

Abstract. A general theory for studying discrete integrable systems is developed based on a trace identity that we previously proposed. A scheme for generating hierarchies of discrete integrable systems is presented. Under broad assumptions the resulting hierarchies are shown to consist of Liouville integrable Hamiltonian systems.

1. Introduction

The theory of continuous integrable systems has been extensively and actively developed in the past twenty years. As compared with the continuous case, the study of discrete integrable systems (see e.g. [1-4] and references therein) has received relatively less attention.

In a recent paper [1], Ragnisco and Santini developed a unified algebraic approach to discrete integrable equations. They studied the discrete spectral problem

$$E\psi = (Q + \lambda A)\psi \tag{1.1}$$

where E is a translation operator defined on a suitable function space, Q is the field matrix, A is a constant invertible matrix, λ is the spectral parameter. By using the theory of bi-Hamiltonian systems [5, 6], they succeeded in establishing the Hamiltonian structure of the hierarchy of nonlinear evolution equations connected to the above spectral problem (1.1).

In another recent paper [2], Schilling proposed a systematic approach to the soliton equations connected with the discrete Ablowitz-Ladik spectral problem

$$p(n)f(n+1) = (E_z + q(n))f(n) + r(n)f(n-1)$$

where $p(n)$, $q(n)$ and $r(n)$ are 2×2 matrices, $E_z = \operatorname{diag}(z, z^{-1})$, and z is the spectral parameter.

In a monograph [7] Kupershmidt developed a theory of discrete formal variational calculus and studied the discrete integrable systems that are connected to the Lax equation

$$(E^p + u_1 E^{p-1} + \ldots + u_p)y = \lambda y \tag{1.2}$$

where $u_1, \ldots, u_p$ are scalar field elements.

All of the above three spectral problems are special cases of the following general problem:

$$E\psi = U\psi \tag{1.3}$$

where $U = U(u, \lambda)$ is an $N \times N$ matrix depending on the field vector $u = (u_1, \ldots, u_p)^{\mathrm{T}}$ and the spectral parameter λ.

One incredible advantage, among others, of the beautiful theory of bi-Hamiltonian systems lies in the fact that the theory can be developed independently without invoking spectral problems. However, the calculation required to verify some coupling conditions is usually very lengthy. The aim of the present paper is to show that the trace identity that we proposed in [8, 9] applies equally well to the discrete case (1.3). By making use of the trace identity, both the hierarchy of equations and the Hamiltonians can be simultaneously derived from a single equation. This method, which makes use of the spectral problem, needs less calculation and can be applied to the systems that are not bi-Hamiltonian. We also propose a scheme for generating the discrete integrable systems connected to (1.3), and establish the Liouville integrability of the resulting Hamiltonian systems.

The paper divides into seven sections. The next section contains a brief presentation on discrete Hamiltonian systems. Then section 3 proposes a scheme for generating discrete integrable systems. As an illustrative example of the general scheme, we establish in section 4 the Hamiltonian structure of the famous Toda lattice hierarchy. The main theorems regarding a trace identity and a general formula for Poisson brackets are presented, respectively, in section 5 and section 6. Section 7 presents a technique for proving the locality of discrete integrable systems.

2. Discrete Hamiltonian systems

This section contain a brief presentation on discrete Hamiltonian systems. The reader is referred to [7] (chapters 2 and 3) for full details.

Let $u = (u_1, \ldots, u_p)^{\mathrm{T}}$ be a vector with the components $u_i = u_i(n, t)$ depending on integers $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. We define the translation and difference operators by

$$(Ef)(n) \equiv f(n+1) \qquad (Df)(n) \equiv f(n+1) - f(n) = (E-1)f(n). \tag{2.1}$$

Throughout this paper we write $f^{(k)} = E^k f$.

For a given (scalar or vector) function $f = f(n)$, if we could find a function h such that $f = Dh$ then we could write $f \sim 0 \pmod D$, i.e.

$$f \sim 0 \pmod D \Leftrightarrow \exists h \qquad \text{such that} \qquad f = Dh.$$

In this case we say that f is equivalent to zero. When no confusion would arise we simply write $f \sim 0$ to mean $f \sim 0 \pmod D$. If the difference of two functions f and g is equivalent to zero, then we write $f \sim g$, i.e.

$$f \sim g \Leftrightarrow f - g \sim 0.$$

For example

$$f \sim f^{(1)} \tag{2.2}$$

since $f^{(1)} - f = Df \sim 0$.

For a scalar function $f = f(u)$, its gradient $(\nabla f)(u)$ is defined in the usual way by

$$(\mathrm{d}/\mathrm{d}\varepsilon) f(u + \varepsilon v)|_{\varepsilon=0} = (\nabla f, v) \tag{2.3}$$

where $v = (v_1, \ldots, v_p)^{\mathrm{T}}$, $\nabla f = ((\nabla f)_1, \ldots, (\nabla f)_p)$, and

$$(f, g) = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{p} f_i(n) g_i(n).$$

It is easy to see that the ith component $(\nabla f)_i$ is given by

$$(\nabla f)_i = \frac{\delta f}{\delta u_i} \equiv \sum_{n\in\mathbb{Z}} E^{-n}\left(\frac{\partial f}{\partial u_i^{(n)}}\right). \tag{2.4}$$

We call $\delta f/\delta u_i$ the discrete variational derivatives. It is known that [7]

$$(\delta/\delta u_i)D = 0. \tag{2.5a}$$

In particular, we have

$$(\delta/\delta u_i)f^{(1)} = (\delta/\delta u_i)f. \tag{2.5b}$$

A linear operator J, mapping the space of p-vectors into itself, is called a Hamiltonian operator if for any two scalar functions f and g the expression

$$\{f, g\} \equiv (J\nabla f, \nabla g) = \sum_{m\in\mathbb{Z}} \sum_{i=1}^{p} \left(J\frac{\delta f}{\delta u}(n)\right)_i \left(\frac{\delta g}{\delta u_i}(n)\right)_i \tag{2.6}$$

is a well defined Poisson bracket, i.e.

$$\{f, g\} = -\{g, f\} \qquad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

In this case we call the evolution equation

$$u_t = \frac{J\delta H(u)}{\delta u} \tag{2.7}$$

a discrete Hamiltonian system and call the scalar function $H(u)$ the Hamiltonian. This system is called Liouville integrable if there exist an infinite number of conserved quantities H_n, $n = 0, 1, 2, \ldots$ such that they are involution in pairs, i.e.

$$(H_n)_t = 0 \tag{2.8}$$

$$\{H_n, H_m\} = 0. \tag{2.9}$$

Usually we take $H = H_{n_0}$ for some $n_0 \in \mathbb{Z}$, then (2.8) is a consequence of (2.9):

$$(H_n)_t = \{H_{n_0}, H_n\} = 0.$$

One of the central concerns in the theory of discrete Hamiltonian systems is to generate as many Liouville integrable discrete Hamiltonian systems as possible. An effective way to do it is to make use of discrete spectral problems, which we shall do in the next section.

3. A scheme for generating discrete integrable systems

Throughout this paper we shall consider the following discrete spectral problem:

$$E\psi = U\psi \tag{3.1}$$

where $\psi = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}$ is an N-vector and $U = U(u, t, \lambda)$ is an $N \times N$ matrix depending on a field vector $u = (u_1, \ldots, u_p)^{\mathrm{T}}$, the time variable t, and a spectral parameter λ. We call (3.1) an isospectral problem if $\lambda_t = 0$.

As in the continuous case, we shall combine equation (3.1) with its t-evolution part

$$\psi_t = \bar{V}\psi \tag{3.2}$$

for some matrix $\bar{V}$. The compatibility condition between (3.1) and (3.2) gives a differential–difference equation

$$U_t = (E\bar{V})U - U\bar{V} \tag{3.3}$$

which will be called a discrete zero-curvature equation. Here the term 'discrete zero-curvature equation' will remind us of the counterpart 'zero-curvature equation' in the continuous case

$$U_t = \bar{V}_x - [U, \bar{V}]$$

which is the compatability between (3.2) and

$$\psi_x = U\psi.$$

The aim of the present paper is to show that for a properly chosen isospectral problem (3.1) we can relate it to a hierarchy of equations

$$\psi_t = V_{(n)}\psi$$

such that the corresponding discrete zero-curvature equations

$$U_{t_n} = (EV_{(n)})U - UV_{(n)} \tag{3.4}$$

are Liouville integrable. To be more precise, we shall show that there exists a common set of conserved quantities $\{H_n\}$ such that (3.4) are equivalent to their Hamiltonian forms

$$u_{t_n} = \frac{J\delta H_n}{\delta u}$$

and the Hamiltonians $\{H_n\}$ are in involution in pairs

$$\{H_n, H_m\} = 0.$$

Let G be a finite-dimensional Lie algebra over $\mathbb{C}$, and let $\tilde{G}$ be the corresponding loop algebra

$$\tilde{G} = G \otimes C(\lambda, \lambda^{-1})$$

where $C(\lambda, \lambda^{-1})$ is the set of Laurent polynomials in λ. Throughout this paper we consider the isospectral problem (3.1) with the matrix

$$U = e_0 + u_1 e_1 + \ldots + u_p e_p \tag{3.5}$$

where $u_i = u_i(n, t)$, $i = 1, \ldots, p$, are field variables depending on $n \in \mathbb{Z}$ (the set of integers) and $t \in \mathbb{R}$, and $e_i = e_i(\lambda) \in G$, $i = 0, 1, \ldots, p$.

The aim of the present paper is to propose a general scheme for generating hierarchies of discrete Hamiltonian systems which are Liouville integrable. Starting from a properly chosen isospectral problem (3.1) with U being of the form (3.5), we search for a sequence of auxiliary problems

$$\psi_t = V_{(n)}\psi$$

such that the corresponding discrete zero-curvature equations

$$U_{t_n} = (EV_{(n)})U - UV_{(n)}$$

represent a hierarchy of Liouville integrable Hamiltonian systems.

The scheme consists of four main steps. First we solve for Γ from the following equation:

$$(E\Gamma)U - U\Gamma = 0. \tag{3.6}$$

We observe that the above equation can be obtained from (3.3) by assuming $U_t = 0$. As in the continuous case, we call the above equation the stationary zero-curvature equation. By substituting the expansion

$$\Gamma = \sum_{i\geq 0} \Gamma_i \lambda^{-i}$$

into (3.6) we obtain a recurrence relation among the Γ_i from which we could calculate Γ_i recurrently.

Second, we take the positive part $(\lambda^n\Gamma)_+$ of $\lambda^n\Gamma$. The positive part f_+ and the negative part f_- of a function $f = \Sigma_i f_i\lambda^i$ are defined by

$$f_+ = \left(\sum_{i\in\mathbb{Z}} f_i\lambda^i\right)_+ \equiv \sum_{i\geq\pi} f_i\lambda^i \qquad f_- \equiv f - f_+$$

where π is a fixed integer. Usually we take $\pi = 0$, in this case we have

$$(\lambda^n\Gamma)_+ = \left(\sum_{i\geq 0} \Gamma_i\lambda^{n-i}\right)_+ \equiv \sum_{i=0}^{n} \Gamma_i\lambda^{n-i}.$$

Then we check if the expression

$$(E(\lambda^n\Gamma)_+)U - U(\lambda^n\Gamma)_+$$

is compatible with U_t, i.e. we check if the condition

$$(E(\lambda^n\Gamma)_+)U - U(\lambda^n\Gamma)_+ \in \sum_{i=1}^{p} \mathbb{C}e_i \tag{3.7}$$

is satisfied. If the above condition holds then we obtain the following hierarchy of equations:

$$U_{t_n} = (EV^{(n)})U - UV^{(n)} \tag{3.8}$$

where $V^{(n)} = (\lambda^n V)_+$. If the condition (3.7) does not hold then we try to find a sequence of modification matrices Δ_n such that for

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n$$

we have

$$(EV^{(n)})U - U(V^{(n)}) \equiv \sum_{i=1}^{p} f_{in} \in \sum_{i=1}^{p} \mathbb{C}_{e_i}.$$

Once we find such a sequence of matrices Δ_n, the corresponding hierarchy of equations will be given by (3.8), or equivalently by

$$u_{t_n} = K_n(u) \tag{3.9}$$

where $K_n(u) = (f_{1_n}(u), \ldots, f_{pn}(u))^{\mathrm{T}}$.

The third step is to rewrite equation (3.9) in its Hamiltonian form

$$u_{t_n} = \frac{J\delta H_n}{\delta u}$$

for a sequence of Hamiltonians $\{H_n\}$. To do so we have to apply the following trace identity:

$$\frac{\delta}{\delta u_i}\langle V, U_\lambda\rangle = \left[\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right]\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle \tag{3.10}$$

where

$$\langle A, B\rangle \equiv \mathrm{Tr}(AB)$$

denotes the trace of the product of matrices A and B, and γ is a constant to be fixed each time; V is defined in terms of Γ:

$$\Gamma = VU. \tag{3.11}$$

The proof of this trace identity will be given in section 5. After using this identity we obtain a sequence of variational relations

$$\frac{\delta H_n}{\delta u} = P_n(u).$$

In most cases we are able to find a Hamiltonian operator J which maps P_n to K_n:

$$K_n(u) = JP_n(u).$$

Then the above hierarchy of equations (3.9) takes its Hamiltonian form

$$u_{t_n} = J\frac{\delta H_n}{\delta u}.$$

The final step is to show the involutive property represented by equation (2.9). This can be done by using a general formula on Poisson brackets:

$$\mu^k\{H(\lambda), H(\mu)\} = D\left\langle\Gamma(\lambda), \frac{\mu}{\mu-\lambda}\Gamma(\mu) + \Delta(\lambda, \mu)\right\rangle \tag{3.12}$$

where k is an integer determined by the Hamiltonian operator J,

$$H(\lambda) = \sum_{n\geqslant 0} H_n\lambda^{-n}$$

and

$$\Delta(\lambda, \mu) = \sum_{n\geqslant 0} \Delta_n(\lambda)\mu^{-n}.$$

The proof of the above formula for Poisson brackets will be given in section 6.

Remark 1. The proof of the trace identity makes use of the notion of 'rank' as used in the continuous case [10]. The rank is an integer-valued function which is defined on u, λ, $e \in \mathrm{G}$ and E such that

$$\rho(ab) = \rho(a) + \rho(b)$$

when ab makes sense, where $\rho(a)$ represents the rank of a. For example, if we have defined $\rho(u_1)$ and $\rho(e_1)$ then

$$\rho(u_1e_1) = \rho(u_1) + \rho(e_1).$$

There is a difference between discrete and continuous cases. In the continuous case the stationary zero-curvature equation is

$$V_x = [U, V]$$

it contains a commutator; while in the discrete case the corresponding stationary zero-curvature equation is (3.6), which does not contain a commutator. Thus we could relax somewhat the condition on ranks by not insisting on the condition $\rho([e,f]) = \rho(e)+\rho(f)$ for $e, f \in G$. We shall define the rank of e_i and u_i in such a way that $U = e_0 + u_1 e_1 + \ldots + u_p e_p$ is homogeneous, i.e. each term is of the same rank:

$$\rho(e_0) = \rho(u_1 e_1) = \ldots = \rho(u_p e_p)$$

or equivalently

$$\rho(u_i) = \rho(e_0) - \rho(e_i) \qquad i = 1, \ldots, p.$$

As distinct from the continuous case, where $\rho(\partial/\partial x)$ could be non-zero, in the discrete case we see from the equation (3.6) that we always have to define

$$\rho(E) = 0$$

in order to keep equations homogeneous in ranks. Next, from the spectral problem (3.1) we see that

$$\rho(U) = 0.$$

It remains an open problem as how to introduce the rank in a systematic way as done in the continuous case [10]. However, once we have succeeded in defining the rank we are able to use the trace identity.

The above picture will be more clear in the next section when we follow the above procedure to deal with a known Toda lattice hierarchy.

Remark 2. The matrix V, defined by (3.11), was first introduced by Ragnisco and Santini [1]. As they pointed out, the matrix V possesses the gradient property which is needed for constructing the desired Hamiltonian. From (3.6) we see that the equation for V is given by

$$(EV)(EU) - UV = 0. \tag{3.13}$$

In fact we have

$$\begin{aligned} (EV)(EU)U &= (E(VU))U \\ &= (E\Gamma)U \qquad \text{using (3.11)} \\ &= U\Gamma \qquad \text{using (3.6)} \\ &= (UV)U \end{aligned}$$

from which we reduce (3.13) in the case that U is invertible, which we shall assume from now on.

From (3.11) and (3.13) we see that

$$\begin{aligned} D\Gamma &= D(VU) \\ &= E(VU) - VU \\ &= UV - VU \\ &= [U, V]. \end{aligned}$$

Thus

$$D\Gamma = [U, V] \tag{3.14}$$

from which we deduce that $D(\mathrm{Tr}\,\Gamma) = 0$. In other words, the trace of Γ is independent of the lattice variable n, therefore without losing generality we may assume that

$$\mathrm{Tr}(\Gamma) = 0. \tag{3.15}$$

4. The Toda hierarchy

As an illustrative example, we now apply the general procedure presented above to establish the Hamiltonian structure of the well known Toda lattice hierarchy [7]. The same procedure has also been applied to recover the Hamiltonian structure of the Ablowitz-Ladik heirarchy [3], and to find new discrete integrable systems that will be reported elsewhere [11].

We consider the isospectral problem (3.1) with the matrix

$$U = U(u, \lambda) = \begin{bmatrix} 0 & 1 \\ -v & \lambda - p \end{bmatrix} \tag{4.1}$$

where

$$u = (p, v)^{\mathrm{T}} \tag{4.2}$$

is the field vector. As is easily seen, the spectral problem (3.1) with (4.1) is equivalent to the scalar spectral problem

$$(E + p + vE^{-1})y = \lambda y$$

with $\psi = (E^{-1}y, y)^{\mathrm{T}}$.

To derive the corresponding hierarchy we proceed first to solve the stationary zero-curvature equation (3.6). By (3.15) we can assume that

$$\Gamma = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

It is easy to verify that

$$(E\Gamma)U - U\Gamma = \begin{bmatrix} -vb^{(1)} - c & a^{(1)} + a + (\lambda - p)b^{(1)} \\ v(a + a^{(1)}) - (\lambda - p)c & (\lambda - p)(a - a^{(1)}) + vb + c^{(1)} \end{bmatrix}. \tag{4.3}$$

Thus the equation (3.6) gives

$$c = vb^{(1)} \tag{4.4a}$$

$$\lambda b^{(1)} = pb^{(1)} - (a^{(1)} + a) \tag{4.4b}$$

$$\lambda(a^{(1)} - a) = (vb - v^{(1)}b^{(2)}) + p(a^{(1)} - a). \tag{4.4c}$$

Substitution the expansions

$$a = \sum_{n\geq 0} a_n\lambda^{-n} \qquad b = \sum_{n\geq 0} b_n\lambda^{-n} \qquad c = \sum_{n\geq 0} c_n\lambda^{-n}$$

into (4.4a)-(4.4c), we obtain the recurrence relations

$$c_n = -vb_n^{(1)} \tag{4.5a}$$

$$b_{n+1}^{(1)} = pb_n^{(1)} - (a_n^{(1)} + a_n) \tag{4.5b}$$

$$a_{n+1}^{(1)} - a_{n+1} = p(a_n^{(1)} - a_n) + (vb_n - v^{(1)}b_n^{(2)}) \tag{4.5c}$$

where $a_n^{(1)} = Ea_n$, $b_n^{(2)} = E^2 b_n$, etc, as mentioned in section 2. To solve the above recurrence relations we have to introduce the rank to rule out the constant which appears when solving for a_{n+1} from (4.5*c*). Taking into consideration the fact that $\rho(E)=0$ and $\rho(U)=0$, as mentioned in the previous section, after some calculation we find that we could introduce the ranks as follows:

$$\rho(h_1)=\rho(h_2)=-1 \qquad \rho(e)=0 \qquad \rho(f)=-2$$
$$\rho(E)=0 \qquad \rho(\lambda)=1$$
$$\rho(a)=0 \qquad \rho(b)=-1 \qquad \rho(c)=1$$
$$\rho(p)=1 \qquad \rho(v)=2 \qquad \rho(\alpha)=0$$

where $\alpha=$ constant, and

$$h_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \qquad h_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad e=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \qquad f=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

Then

$$\rho(U)=\rho((e+\lambda h_2)-vf-ph_2)=0$$
$$\rho(\Gamma)=\rho(a(h_1-h_2)+be+cf)=-1$$
$$\rho(V)=\rho(\Gamma U^{-1})=-1$$

and

$$\rho(a_n)=\rho(a_n\lambda^{-n})-\rho(\lambda^{-n})=\rho(a)+n\rho(\lambda)=n$$
$$\rho(b_n)=\rho(b)+n\rho(\lambda)=n-1.$$

The requirement $\rho(a_0)=\rho(b_1)=0$ forces us to take $a_0=$ constant, $b_1=$ constant, thus we can take the initial values as

$$a_0=\tfrac{1}{2} \qquad b_0=0 \qquad b_1=-1. \tag{4.7}$$

Starting from the above initial values and using the recurrence relations (4.5*b*) and (4.5*c*), we can easily calculate the first few a_n and b_n as follows:

$$a_1=0 \qquad a_2=v \qquad a_3=v(p+p^{(-1)}) \tag{4.8a}$$
$$a_4=v(p^2+pp^{(-1)}+p^{(-1)^2}+v^{(-1)}+v+v^{(1)}) \tag{4.8b}$$
$$a_5=v(p^3+p^2p^{(-1)}+pp^{(-1)^2}+pv^{(-1)}+p^{(1)}v^{(1)}+2vp+p^{(-1)^3}+2vp^{(-1)}$$
$$+v^{(-1)}p^{(-2)}+v^{(1)}p^{(-1)}+2v^{(-1)}p^{(-1)}+2v^{(1)}p) \tag{4.8c}$$
$$b_2=p^{(-1)} \qquad b_3=-(v^{(-1)}+v+p^{(-1)^2}) \tag{4.8d}$$
$$b_4=-(vp+2vp^{(-1)}+2v^{(-1)}p^{(-1)}+v^{(-1)}p^{(-2)}+p^{(-1)^3}) \tag{4.8e}$$
$$b_5=-(2v^{(-1)}v+v^2+vv^{(1)}+v^{(-1)}v^{(-2)}+v^{(-1)^2}+p^{(-1)^4}+vp^2+2pp^{(-1)}v+3vp^{(-1)^2}$$
$$+3v^{(-1)}p^{(-1)^2}+2v^{(-1)}p^{(-1)}p^{(-2)}+v^{(-1)}p^{(-2)^2}). \tag{4.8f}$$

It is worthwhile observing that the recurrence relation (4.5*c*) is not local in the sense that once we have calculated the expression $A_n \equiv p(a_n^{(1)}-a_n)+(vb_n-v^{(1)}b_n^{(1)})$ we have to perform the integration D^{-1} in order to find $a_{n+1}=D^{-1}A_n$. Thus only in the case that

$$p(a_n^{(1)}-a_n)+(vb_n-v^{(1)}b_n^{(1)})\sim 0 \tag{4.9}$$

can we obtain a local expression for a_{n+1}, i.e. an expression involving only a finite number of $v^{(i)}$ and $p^{(i)}$. As we have seen above, equation (4.9) does hold for the first few n. We show in section 7 that (4.9) does hold for all n. In other words, the Toda hierarchy consists of local systems.

Now we turn to searching for the corresponding hierarchy. By substituting $(\lambda^n\Gamma) = (\lambda^n\Gamma)_+ + (\lambda^n\Gamma)_-$ into (3.6) we find that

$$(E(\lambda^n\Gamma)_+)U - U(\lambda^n\Gamma)_+ = U(\lambda^n\Gamma)_- - (E(\lambda^n\Gamma)_-)U. \tag{4.10}$$

We observe that the left-hand side of (4.10) contains terms with powers λ^k, $k \geqslant 0$, while the right-hand side contains terms with powers λ^k, $k \leqslant 0$. Therefore both sides of (4.10) contains only terms with $\lambda^k|_{k=0}$. In other words, the expression

$$(E(\lambda^n\Gamma)_+)U - U(\lambda^n\Gamma)_+$$

is λ independent. Thus

$$\begin{aligned}(E(\lambda^n\Gamma)_+)U &- U(\lambda^n\Gamma)_+ \\ &= (E(\lambda^n\Gamma)_+)U - U(\lambda^n\Gamma)_+|_{\lambda=0} \\ &= (E\Gamma_n)U_1 - U_1\Gamma_n \\ &= \begin{bmatrix} 0 & -b_{n+1}^{(1)} \\ v(a_n + a_n^{(1)}) + pc_n & vb_n + c_n^{(1)} + p(a_n^{(1)} - a_n) \end{bmatrix}\end{aligned} \tag{4.11}$$

where we have made the expansion

$$\Gamma = \sum_{n\geqslant 0} \Gamma_n \lambda^{-n} \qquad \Gamma_n = \begin{bmatrix} a_n & b_n \\ c_n & -a_n \end{bmatrix} \qquad U = \lambda U_0 + U_1. \tag{4.12}$$

Following the same scheme that we proposed in [12] for generating continuous integrable systems, we search for a modification matrix Δ_n such that for some functions f_n and g_n it holds that

$$(EV_{(n)})U - UV_{(n)} = \begin{bmatrix} 0 & 0 \\ -g_n & -f_n \end{bmatrix} \tag{4.13}$$

with $V_{(n)}$ defined by (3.5). Once the matrix Δ_n has been found, equation (3.4) would give the desired hierarchy.

It is easy to verify that if $\Delta_n = \mathrm{diag}(\alpha_n, 0)$, then

$$(E\Delta_n)U - U\Delta_n = \begin{bmatrix} 0 & \alpha_n^{(1)} \\ v\alpha_n & 0 \end{bmatrix}. \tag{4.14}$$

Thus to cancel the matrix element $-b_{n+1}^{(1)}$ appearing in (4.11) we may set

$$\alpha_n = b_{n+1}.$$

Then by (4.11), (4.13) and (4.14) we obtain the resulting hierarchy (3.4) as

$$p_{t_n} = f_n = (v^{(1)}b_n^{(2)} - vb_n) + p(a_n - a_n^{(1)}) \tag{4.15a}$$

$$v_{t_n} = g_n = v(pb_n^{(1)} - a_n - a_n^{(1)} - b_{n+1}). \tag{4.15b}$$

By using the recurrence relations (4.5b) and (4.5c) we can rewrite the above hierarchy as

$$p_{t_n} = -(a_{n+1}^{(1)} - a_{n+1}) \tag{4.16a}$$

$$v_{t_n} = v(b_{n+1}^{(1)} - b_{n+1}). \tag{4.16b}$$

This is the Toda hierarchy we are searching for.

The first few systems of this hierarchy can be obtained by using (4.8*a*)-(4.8*f*). The first system is

$$p_{t_2} = -(v^{(1)} - v) \qquad v_{t_2} = -v(p - p^{(-1)}).$$

which is just the Toda lattice equation:

$$\mathrm{d}p(n)/\mathrm{d}t = v(n) - v(n+1)$$
$$\mathrm{d}v(n)/\mathrm{d}t = v(n)(p(n-1) - p(n)) \qquad (\mathrm{d}/\mathrm{d}t = \mathrm{d}/\mathrm{d}t_2).$$

The next two systems are as follows:

$$p_{t_3} = v(p + p^{(-1)}) - v^{(1)}(p^{(1)} + p)$$
$$v_{t_3} = v(v_{(-1)} - v^{(1)} + p^{(-1)^2} - p^2)$$
$$p_{t_4} = v(p^2 + pp^{(-1)} + p^{(-1)^2} + v^{(-1)} - v) - v^{(1)}(p^{(1)^2} + pp^{(1)} + p^2 + v^{(1)} + v^{(2)})$$
$$v_{t_4} = v(vp^{(-1)} + 2v^{(-1)}p^{(-1)} + v^{(-1)}p^{(-2)} + p^{(-1)^3} - v^{(1)}p^{(1)} - 2v^{(1)}p - vp - p^3).$$

Our next target is to write the Toda hierarchy (4.16*a*) and (4.16*b*) in its Hamiltonian form. To this end we apply the following trace identity:

$$(\delta/\delta u_i)\langle V, U_\lambda\rangle = (\lambda^{-\gamma}(\partial/\partial\lambda)\lambda^{\gamma})\langle V, \partial U/\partial u_i\rangle \tag{4.17}$$

where

$$\langle A, B\rangle = \mathrm{Tr}(AB)$$

and γ is a constant to be fixed and V is a solution of equation (3.9). The proof of the trace identity (4.17) will be given in the next section.

By (3.8) we have

$$\begin{aligned} V &= \Gamma U^{-1} \\ &= \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} (\lambda - p)/v & -1/v \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} V_{11} & -a/v \\ V_{21} & -c/v \end{bmatrix} \\ &= \begin{bmatrix} V_{11} & -a/v \\ V_{21} & b^{(1)} \end{bmatrix}. \end{aligned}$$

Thus

$$\langle V, U_\lambda\rangle = \left\langle V, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\rangle = b^{(1)}$$
$$\left\langle V, \frac{\partial U}{\partial p}\right\rangle = \left\langle V, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right\rangle = -b^{(1)}$$
$$\left\langle V, \frac{\partial U}{\partial v}\right\rangle = \left\langle V, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right\rangle = \frac{a}{v}.$$

The trace identity (4.17) then gives

$$\frac{\delta b^{(1)}}{\delta u} \equiv \left(\frac{\delta}{\delta p}, \frac{\delta}{\delta v}\right) b^{(1)} = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)\left(-b^{(1)}, \frac{a}{v}\right).$$

Using (2.5*b*) and making an expansion, we obtain

$$\frac{\delta b_{n+1}}{\delta u}=(\gamma-n)\left(-b_n^{(1)},\frac{a_n}{v}\right).$$

To fix the constant γ, we simply set $n=1$ in the above equation and obtain

$$(-1,0)=\left(\frac{\delta}{\delta p},\frac{\delta}{\delta v}\right)(-p^{(-1)})=(\gamma-1)\left(-b_1^{(1)},\frac{a_1}{v}\right)=(\gamma-1)(1,0)$$

hence $\gamma=0$ and we find

$$\left(-b_n^{(1)},\frac{a_n}{v}\right)=\frac{\delta H_n}{\delta u}$$

with

$$H_n=-b_{n+1}/n.$$

Next we search for a matrix J such that

$$J\begin{bmatrix}-b_{n+1}^{(1)}\\ a_{n+1}/v\end{bmatrix}=\begin{bmatrix}f_n\\ g_n\end{bmatrix}=\begin{bmatrix}(a_{n+1}-a_{n+1}^{(1)})\\ v(b_{n+1}^{(1)}-b_{n+1})\end{bmatrix}.$$

The matrix J satisfying the above condition is clearly given by

$$J=\begin{bmatrix}0 & (1-E)v\\ v(E^{-1}-1) & 0\end{bmatrix}$$

which is easily seen to be a Hamiltonian operator. Therefore we have succeeded in writing the Toda hierarchy in the following Hamiltonian form:

$$\begin{aligned}u_{t_n}&=\begin{bmatrix}p\\ v\end{bmatrix}_{t_n}\\ &=\begin{bmatrix}f_n\\ g_n\end{bmatrix}\\ &=J\begin{bmatrix}-b_{n+1}^{(1)}\\ a_{n+1}/v\end{bmatrix}\\ &=J\frac{\delta H_{n+1}}{\delta u}.\end{aligned}$$

Furthermore, by a general result to be presented in the next section, we know that

$$\{H_n,H_m\}=0$$

which shows that the Toda hierarchy consists of completely integrable discrete Hamiltonian systems.

5. The trace identity

We prove in this section the trace identity (4.17).

Since in both the continuous case and the discrete case, the gradient of a functional is defined by the same equation (2.3), thus the technique 'constrained variational calculus (CVC) [8] also applies to the present discrete case. The CVC technique is used to find the variational derivative $\delta F(w)/\delta u$ under the constraint $R(u,w)=0$. The main steps of CVC are as follows [8].

(i) Firstly we introduce the 'Lagrangian multiplier' Λ, which is usually a matrix with the same dimension as the matrix R^{T}, and form the sum

$$\Phi = F + \mathrm{Tr}(\Lambda R).$$

(ii) Secondly we treat u and w as independent variables and set $\delta\Phi/\delta w = 0$.

(iii) Finally, treating u and w as independent variables again, we have

$$\frac{\delta F}{\delta u} = \frac{\hat{\delta}}{\delta u}\Phi$$

where the notation $\hat{\delta}/\delta u$ is used to mean that we calculate $\delta\Phi(u, w)/\delta u$ as if w were independent of u.

To formulate the trace identity we need the notion of rank as explained in the previous section.

Theorem 1 (trace identity). Suppose that the solution of equation (3.9) is unique in the sense that two solutions V_1 and V_2 of the same rank differ only by a constant factor: $V_2 = \alpha V_1$, α = constant. Then it holds that

$$\frac{\delta}{\delta u_i}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\right)\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle \tag{5.1}$$

where γ is a constant and V is a solution of equation (3.9) which is of homogeneous rank.

Proof. We apply the CVC technique to calculate the variational derivative

$$\frac{\delta}{\delta u_i}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle.$$

To this end we introduce the Lagrangian multiplier $N \times N$ matrix Λ and form the sum

$$\Phi = \langle V, U_\lambda\rangle + \langle (EV)(EU) - UV, \Lambda\rangle.$$

That is, we take $F = \langle V, U_\lambda\rangle$, $R = (EV)(EU) - UV$, as mentioned in the above description of the CVC procedure. We have

$$0 = \frac{\delta\Phi}{\delta V^{\mathrm{T}}} = U_\lambda + U(E^{-1}\Lambda) - \Lambda U. \tag{5.2}$$

According to the CVC procedure we then obtain

$$\begin{aligned}\frac{\delta}{\delta u_i}\langle V, U_\lambda\rangle &= \frac{\hat{\delta}}{\delta u_i}\Phi \\ &= \left\langle V, \frac{\partial U_\lambda}{\partial u_i}\right\rangle + \left\langle \frac{\partial U}{\partial u_i}, (E^{-1}\Lambda)V - V\Lambda\right\rangle \end{aligned} \tag{5.3}$$

where we have made use of

$$\begin{aligned}\frac{\hat{\delta}}{\delta u_i}\langle (EV)(EU), \Lambda\rangle &= \frac{\hat{\delta}}{\delta u_i}\langle VU, E^{-1}\Lambda\rangle \\ &= \frac{\hat{\delta}}{\delta u_i}\langle U, (E^{-1}\Lambda)V\rangle \\ &= \left\langle \frac{\partial U}{\partial u_i}, (E^{-1}\Lambda)V\right\rangle.\end{aligned}$$

Now setting $G \equiv (E^{-1}\Lambda)V - V\Lambda$, we find on the one hand that

$$
\begin{aligned}
(EG)(EU) - UG \\
&= \Lambda(EV)(EU) - (EV)(E\Lambda)(EU) - U(E^{-1}\Lambda)V + UV\Lambda \\
&= \Lambda UV - (EV)(E\Lambda)(EU) - U(E^{-1}\Lambda)V + (EV)(EU)\Lambda \qquad \text{using (3.8)} \\
&= (\Lambda U - U(E^{-1}))V + (EV)((EU)\Lambda - (E\Lambda)(EU)) \\
&= U_\lambda V - (EV)(EU_\lambda) \qquad \text{using (5.2).} \qquad (5.4)
\end{aligned}
$$

On the other hand, by differentiating both sides of equation (3.13) with respect to λ, we find

$$(EV_\lambda)(EU) + (EV)(EU_\lambda) = U_\lambda V + UV_\lambda$$

hence

$$(EV_\lambda)(EU) - UV_\lambda = U_\lambda V - (EV)(EU_\lambda). \qquad (5.5)$$

From (5.4) and (5.5) we deduce that

$$(E(G - V_\lambda))(EU) - U(G - V_\lambda) = 0.$$

Thus we see that $G - V_\lambda$ is again a solution of (3.13). Since V/λ is clearly a solution of (3.13) and V/λ has the same rank as $G - V_\lambda$, therefore by the supposition we have

$$G - V_\lambda = (\gamma/\lambda)V \qquad (5.6)$$

with γ a constant. Substituting (5.6) into (5.3), we find

$$
\begin{aligned}
\frac{\delta}{\delta u_i}\langle V, U_\lambda\rangle &= \left\langle V, \frac{\partial U_\lambda}{\partial u_i}\right\rangle + \left\langle \frac{\partial U}{\partial u_i}, G\right\rangle \\
&= \left\langle V, \frac{\partial U_\lambda}{\partial u_i}\right\rangle + \left\langle \frac{\partial U}{\partial u_i}, V_\lambda\right\rangle + \left\langle \frac{\partial U}{\partial u_i}, \left(\frac{\gamma}{\lambda}\right)V\right\rangle \\
&= \left(\frac{\partial}{\partial\lambda} + \frac{\gamma}{\lambda}\right)\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle \\
&= \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle.
\end{aligned}
$$

The proof is completed. □

Remark. It is worthwhile observing that the trace identity in the present discrete case takes completely the same form as in the continuous case [8-10, 13, 14]. There are certainly some differences. The matrix V is different: in the continuous case the matrix V is supposed to be a solution of the equation $V_x = [U, V]$, while in the present discrete case the matrix V is taken to be a solution of the equation $D\Gamma = [U, V]$. Both the choices of V come from the common consideration that they are gradients.

6. Liouville integrability

In the continuous case we have established [15] an explicit formula for Poisson brackets by which we prove that under broad assumptions the hierarchies derived from isospectral problems consist of Liouville integrable Hamiltonian systems. As a consequence of the Liouville integrability, the flows generated by different systems in the same hierarchy are always commutative. The aim of the present section is to do the same in the present discrete case.

Following [15], we shall use the concise notation

$$F_t(\lambda;\mu) \equiv \sum_{n\geq 0} F_{t_n}(\lambda)\mu^{-n} \qquad F_t(\mu) \equiv \sum_{n\geq 0} F_{t_n}\mu^{-n}$$

where F is a matrix or scalar function. Thus we write, for example,

$$U_t(\lambda,\mu) = \sum U_{t_n}(\lambda)\mu^{-n} \qquad (u_i)_t(\mu) = \sum (u_i)_{t_n}\mu^{-n}.$$

We have explained in section 3 the scheme for generating integrable systems related to an isospectral problem

$$E\psi = U\psi \tag{6.1a}$$

with

$$U = e_0 + u_1 e_1 + \ldots + u_p e_p \tag{6.1b}$$

where $e_i = e_i(\lambda)$, $i = 0, 1, \ldots, p$ are matrices belong to some matrix Lie algebra. The main idea of the scheme is to find a matrix

$$\Delta(\lambda,\mu) \equiv \sum_{n\geq 0} \Delta_n(\lambda)\mu^{-n} \tag{6.2}$$

such that

$$(EV_{(n)})U - UV_{(n)} \in \sum_{i=1}^{p} \mathbb{C}_{e_i} \tag{6.3}$$

where

$$V_{(n)} = V_{(n)}(\lambda) \equiv (\lambda^n\Gamma(\lambda))_+ + \Delta_n(\lambda).$$

When the requirement (6.3) is met, i.e.

$$(EV_{(n)})U - UV_{(n)} = \sum_{i=1}^{p} f_{in} e_i$$

holds for a set of functions $\{f_{in}\}$, then the desired hierarchy of equations will be

$$(u_1, \ldots, u_p)_{t_n} \equiv (f_{n1}, \ldots, f_{np})$$

or

$$\begin{aligned} U_{t_n}(\lambda) &= \sum (u_i)_{t_n} e_i(\lambda) \\ &= \sum f_{in} e_i(\lambda) \\ &= (EV_{(n)})U - UV(n). \end{aligned} \tag{6.4}$$

By using the equation [15]

$$\sum_{n\geq 0} (\lambda^n\Gamma(\lambda))_+\mu^{-n} = \frac{\mu}{\mu-\lambda}\Gamma(\mu)$$

we can rewrite the hierarchy (6.4) in the following concise form:

$$\begin{aligned} U_t(\lambda,\mu) &\equiv \sum_{n\geq 0} U_{t_n}(\lambda)\mu^{-n} \\ &= \sum_{n\geq 0} ((EV_{(n)}(\lambda))U(\lambda) - U(\lambda)V_{(n)}(\lambda))\mu^{-n} \\ &= E\left(\frac{\mu}{\mu-\lambda}\Gamma(\mu) + \Delta(\lambda;\mu)\right)U(\lambda) - U(\lambda)\left(\frac{\mu}{\mu-\lambda}\Gamma(\mu) + \Delta(\lambda,\mu)\right). \end{aligned} \tag{6.5}$$

Since

$$E\Gamma(\mu)U(\lambda)-U(\lambda)\Gamma(\mu)$$
$$=E\Gamma(\mu)U(\mu)+E\Gamma(\mu)(U(\lambda)-U(\mu))-U(\lambda)\Gamma(\mu)$$
$$=U(\mu)\Gamma(\mu)-U(\lambda)\Gamma(\mu)+E\Gamma(\mu)(U(\lambda)-U(\mu))$$
$$=(U(\mu)-U(\lambda))\Gamma(\mu)-E\Gamma(\mu)(U(\mu)-U(\lambda))$$

we see that the resulting hierarchy takes the form

$$U_t(\lambda,\mu)=\left(\mu\frac{U(\mu)-U(\lambda)}{\mu-\lambda}\right)\Gamma(\mu)-E\Gamma(\mu)\left(\mu\frac{U(\mu)-U(\lambda)}{\mu-\lambda}\right)$$
$$+E\Delta(\lambda;\mu)U(\lambda)-U(\lambda)\Delta(\lambda;\mu). \tag{6.6}$$

We need the following simple proposition.

Proposition 2. Let V be a solution of equation (3.9) and $U_\tau\equiv \mathrm{d}U(\lambda,u(\tau))/\mathrm{d}\tau$ be defined by

$$U_\tau=(E\bar{V})U-U\bar{V}$$

for some matrix $\bar{V}$. Then

$$\sum_i\left\langle V,\frac{\partial U}{\partial u_i}\right\rangle u_{i\tau}=D\langle\Gamma,\bar{F}\rangle\sim 0\ (\mathrm{mod}\ D).$$

Proof. We have

$$\sum_i\left\langle V,\frac{\partial U}{\partial u_i}\right\rangle u_{i\tau}=\langle V,U_\tau\rangle$$
$$=\langle V,(E\bar{V})U\rangle-\langle V,U\bar{V}\rangle$$
$$=\langle UV,E\bar{V}\rangle-\langle VU,\bar{V}\rangle$$
$$=\langle(EU)(EV),E\bar{V}\rangle-\langle VU,\bar{V}\rangle$$
$$=D\langle VU,\bar{V}\rangle$$
$$=D\langle\Gamma,\bar{V}\rangle\sim 0.$$

The proof is completed. □

Now we are in a position to formulate the main result of this paper as follows.

Theorem 3. Let an isospectral problem be given by (6.1a) and (6.1b). Suppose that:
(i) there exists a matrix $\Delta=\Delta(\lambda;\mu)$ such that the equation

$$\left(\mu\frac{U(\mu)-U(\lambda)}{\mu-\lambda}\right)\Gamma(\mu)-E\Gamma(\mu)\left(\mu\frac{U(\mu)-U(\lambda)}{\mu-\lambda}\right)+E\Delta(\lambda;\mu)U(\lambda)-U(\lambda)\Delta(\lambda;\mu)$$
$$=\sum_{i=1}^{p}f_i(\mu)e_i(\lambda) \tag{6.7}$$

holds for a set of scalar functions $\{f_i\}$;

(ii) there exist a Hamiltonian operator J such that the equation

$$\lambda^k J\left(\left\langle V, \frac{\partial U}{\partial u_1}\right\rangle, \ldots, \left\langle V, \frac{\partial U}{\partial u_p}\right\rangle\right)^{\mathrm{T}} = (f_1, \ldots, f_p)^{\mathrm{T}} \tag{6.8}$$

holds for some integer k.

(a) *(Generation of hierarchy)* A hierarchy of equations can be related to the isospectral problem (6.1a) and (6.1b) which takes the form (6.6), i.e.

$$u_i(\mu)_t = f_i(\mu) \qquad i = 1, \ldots, p \tag{6.9}$$

or equivalently

$$(u_i)_{t_n} = f_{in} \qquad i = 1, \ldots, p. \tag{6.10}$$

(b) *(Derivation of Hamiltonian structure)* Equations (6.10) take the following Hamiltonian form:

$$u_{t_n} = J\frac{\delta H_{n+k}}{\delta u} \tag{6.11}$$

or equivalently

$$u_t = J\frac{\delta \lambda^k H(\mu)}{\delta u} \tag{6.12}$$

where $H(\lambda) \equiv \Sigma_{n\geqslant 0} H_n \lambda^{-n}$ is defined by

$$\left(\lambda^{-\gamma}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\right)H(\lambda) = \left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle. \tag{6.13}$$

(c) *(Liouville integrability)* Each equation in the hierarchy (6.11) is Liouville integrable and the set $\{H_n\}$ constitutes the common set of infinitely many conserved quantities which are in involution in pairs:

$$\{H_m, H_n\} = 0 \qquad \forall m, n \geqslant 0.$$

(d) *(Formula for the Poisson bracket)* Let $H(\lambda)$ be the generating function of $\{H_n\}$, then we have

$$\mu^k\{H(\mu), H(\lambda)\} = \sum_n D\left\langle \Gamma(\lambda), \frac{\mu}{\mu - \lambda}\Gamma(\mu) + \Delta(\lambda, \mu)\right\rangle(n) = 0. \tag{6.14}$$

Proof. The conclusion (a) has been drawn in the beginning of this section. The conclusion (b) can be easily derived from (6.6), (6.7), (6.10) and the trace identity (5.1). In fact the trace identity (5.1) can be decoupled into (6.13) and

$$\frac{\delta H(\lambda)}{\delta u_i} = \left\langle V, \frac{\partial U}{\partial u_i}\right\rangle. \tag{6.15}$$

Since the conclusion (c) is an immediate consequence of (d), it remains to prove (6.14). We have

$$\mu^k\{H(\mu), H(\lambda)\}$$

$$=\sum_{in} \left(\mu^k J \frac{\delta H(\mu)}{\delta u}\right)_i \left(\frac{\delta H(\lambda)}{\delta u_i}\right)(n)$$

$$=\sum_{in} \left(\left(\frac{\delta H(\lambda)}{\delta u}\right)_i u_{it}(\mu)\right)(n) \qquad \text{using (6.11)}$$

$$=\sum_{in} \left(\left\langle V(\lambda), \frac{\partial U(\lambda)}{\partial u_i}\right\rangle u_{it}(\mu)\right)(n) \qquad \text{using (6.15)}$$

$$=\sum_{n} D\left\langle \Gamma(\lambda), \frac{\mu}{\mu-\lambda}\Gamma(\mu)+\Delta(\lambda,\mu)\right\rangle(n) \qquad \text{by (6.5) and proposition 2}$$

$$=0.$$

The proof is completed. □

We observe that the expression (6.14) for Poisson brackets takes essentially the same form as in the continuous case [15].

7. First integrals and the locality of discrete integrable systems

In this section we show that the Toda hierarchy (4.16*a*) and (4.16*b*) consists of local systems. In other words, all a_n and b_n involves only a finite number of $p^{(i)}$ and $v^{(j)}$. In fact, we show that all a_n and b_n are polynomials of $p^{(i)}$ and $v^{(j)}$.

We may recall that in the continuous case we have met similar phenomena: in spite of the fact that some recurrence relations are not local, their solutions are always local. We had suggested in [15] a 'locality lemma' to prove this phenomenon in many cases. The basic idea is to search for some first integrals of the stationary zero-curvature equation. In the continuous case the stationary zero-curvature equation is $V_x=[U, V]$, it admits a number of integrals

$$\alpha_n = \mathrm{Tr}\ V^n$$

for which we have $(\alpha_n)_x=0$. When U is a 2×2 matrix, one integral $\alpha_2=\mathrm{Tr}(V^2)$ would be sufficient to this end [15]. We note that when $V=a(h_1+h_2)+be+cf$ we have

$$\mathrm{Tr}(V^2) = -2\det(V)$$

this first integral was also used by Schilling [2] to show locality.

We show in the present discrete case that a similar result holds.

Proposition 4. Let Γ be defined by equation (3.6). Then we have

$$D(\mathrm{Tr}(\Gamma^n))=0. \tag{7.1}$$

Proof. We have from (3.8) and (3.9) that

$$\Gamma = VU \qquad \Gamma^{(1)} = UV$$

thus

$$\begin{aligned} D(\mathrm{Tr}(\Gamma^n)) &= \mathrm{Tr}(\Gamma^{(1)^n}) - \mathrm{Tr}(\Gamma^n) \\ &= \mathrm{Tr}((UV)^n) - \mathrm{Tr}((VU)^n) \\ &= 0. \end{aligned}$$

The proof is completed. □

The following lemma shows that when U is a 2×2 matrix, one first integral $\alpha_2 = \mathrm{Tr}(\Gamma^2)$ would be sufficient to show the locality.

Proposition 5. Let U be a 2×2 matrix and Γ be defined by (4.12). If b_0, c_0 are local, $a_0 = \alpha = \text{constant} \neq 0$, and suppose that there is a set of recurrence relations

$$\begin{aligned} b_{n+1} &= f_1(a_n, b_n, c_n, a_{n-1}, b_{n-1}, c_{n-1}, \ldots) \\ c_{n+1} &= f_2(a_n, b_n, c_n, a_{n-1}, b_{n-1}, c_{n-1}, \ldots) \end{aligned} \tag{7.2}$$

where f_1 and f_2 are two polynomials of a_i, b_i, c_i, $i \leqslant n$, and their translations, then all a_n, b_n, c_n $(n \geqslant 0)$ are local, that is they depend on a finite number of $u^{(j)}$, where u is the field variable contained in the matrix U.

Proof. By proposition 4 we have

$$a^2 + bc = \tfrac{1}{2}\mathrm{Tr}(\Gamma^2) = \gamma = \text{constant}.$$

Thus

$$a_{n+1} = \frac{-1}{(2\alpha)}\left(\sum_{i=1}^{n} a_i a_{n+1-i} + \sum_{j=0}^{n+1} b_j c_{n+1-j} - \gamma\right). \tag{7.3}$$

The lemma is thus proved by induction on n based on the above equations (7.2) and (7.3). □

Corollary 6. The Toda heirarchy is local.

We could expect that the same technique could be also applied to prove the locality of other discrete integrable systems, for example the system discussed by Bruschi and Ragnisco [16], where they made a similar conjecture on locality.

In subsequent papers we shall apply the method developed here to discuss other integrable systems relating to a variety of discrete isospectral problems.

Acknowledgments

This paper was completed when I was visiting the Department of Physics, Rome University. I am much indebted to Professor Degasperis for his kind invitation and to the department for their warm hospitality. I am also very grateful to Professor Calogero and to Professors Levi, Ragnisco, Santini and Sym for helpful discussions. I would like to express my special thanks to the referee for his valuable suggestion which has been followed in the present improved version of the paper.

This work was supported by the Istituto Nazionale di Fisica Nucleare Italia and the National Natural Science Foundation of China.

References

[1] Ragnisco O and Santini P M 1989 A unified algebraic approach to the integrable and discrete evolution equations *Preprint N.668* Department of Physics, Rome University

[2] Schilling R J 1989 *J. Math. Phys.* **30** 1487-1501

[3] Albowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[4] Fadeeev L D and Takhtajan 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)

[5] Magri F J 1978 *J. Math. Phys.* **19** 1156-62

[6] Fuchssteiner B and Fokas A S 1981 *Physica* **4D** 47

[7] Kupershmidt B A 1985 Discrete Lax equations and differential-difference calculus *Astérique* **123**

[8] Tu G Z 1986 *Scientia Sinica* **24A** 138-48

[9] Tu G Z 1987 *Advances in Science of China (Math.)* **2** 45-72

[10] Tu G Z 1989 *J. Math. Phys.* **30** 330-8

[11] Ragnisco O and Tu G Z 1989 On a new hierarchy of discrete integrable systems *Preprint* Department of Physics, Rome University

[12] Tu G Z 1989 *Science in China* A **32** 142-53

[13] Tu G Z and Men D Z 1989 *Acta Math. Appl. Sinica* (English Series) **5** 89-96

[14] Tu G Z and Xu B Z 1989 The trace identity, a powerful tool for constructing the Hamiltonian structures of integrable systems III *Preprint* Computing Centre, Academia Sinica

[15] Tu G Z 1989 On Liouville integrability of zero-curvature equations and the Yang hierarchy *J. Phys. A: Math. Gen.* **22** 2375-92

[16] Bruschi M and Ragnisco O 1983 *J. Math. Phys.* **24** 1414-21